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NEW FORM OF NJÅSTAD'S α-SET AND LEVINE'S SEMI-OPEN SET

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Abstract. This paper gives an extensive study of ideal topological space and introduce two new types of set with the help of local function. Several characterizations of these sets will also be discussed through this paper and finally gives new representation of α-sets and semi-open sets.

1. Introduction

The study of a-open sets in topological space was introduced by Ekici in [4]. Further he has also studied this type of sets at various aspect in his paper [4, 5, 6, 7]. One of the most important property of this set is, collection of all a-open sets in a topological space (X, τ) forms a topology and it is denoted as τ^a. Most recently the Authors Al-Omeri *et al.* in [2, 1] have introduced ideal on this topological space however the ideal on topological space was first introduced by Kuratowski [13] and Vaidyanathaswamy [18] and it is called ideal topological space. Local function [9, 12, 3, 10, 16] in ideal topological space has been taken an important role for further study of ideal topological space.

Through this paper, we will consider Al-Omeri *et al.*'s ideal topological space and give some representations of Njåstad's α-open set [15] and Levine's semi-open set [14] with the help of $\Re_a$ and $()^{a^*}$ operators. For this job, we shall take the help of two new types of sets as a mathematical tool, one of them is ${\Re_a}^{a^*}$ set and second one is ${\dot{\Re}_a}^{a^*}$. We also discuss several characterizations related to these two sets.

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2. Preliminaries

Let (X,τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X,τ), $Cl(A)$ and $Int(A)$ denote the closure and interior of A in (X,τ), respectively. A subset A of X is said to be regular open [17] (resp. semi-open [14, 8], α-open [15]) if $A = Int(Cl(A))$ (resp. $A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(Int(A)))$. A subset A of X is called δ-open [19] if, for each $x \in A$, there exists a regular open set G such that $x \in G \subseteq A$. The family of all α-open (resp. semi-open) sets in a topological space (X,τ) is denoted by τ^α (resp. $SO(X,\tau)$). The complement of δ-open set is called δ-closed. A point $x \in X$ is called a δ-cluster point of A if $Int(Cl(V)) \cap A \neq \emptyset$, for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and it is denoted by $Cl_\delta(A)$ [19]. The set δ-interior of A is the union of all regular open sets of X contained in A and it is denoted by $Int_\delta(A)$ [19]. A is δ-open if $Int_\delta(A) = A$, δ-open sets form a topology τ^δ.

A subset A of (X,τ) is said to be a-open (resp. a-closed) [5, 6, 7] if $A \subseteq Int(Cl(Int_\delta(A)))$ (resp. $Cl(Int(Cl_\delta(A))) \subseteq A$). The family of all a-open sets of X form a topology on X. This collection is denoted by τ^a [5], and $\tau^a(x)$ is denoted as the collection of all a-open sets containing $x \in X$. If A is a subset of a topological space (X,τ), then the intersection of all a-closed sets containing A is called the a-closure of A and is denoted by $aCl(A)$ [4]. The a-interior of A, denoted by $aInt(A)$, is defined by the union of all a-open sets contained in A [4].

A collection $\mathcal{I} \subseteq \wp(X)$ is said to be an ideal [13] on X if $B \subseteq A \in \mathcal{I}$ implies $B \in \mathcal{I}$ and $A,\ B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Let $\mathcal{I}$ be an ideal on the topological space (X,τ), then $(X,\tau,\mathcal{I})$ is called an ideal topological space. Two operators $()^{a^*}$ and $\Re_a$ have been introduced and studied by Al-Omeri *et al.* [1, 2] in ideal topological spaces. These two operators are defined by the following way:

For a subset A of an ideal topological space $(X,\tau,\mathcal{I})$, $A^{a^*} = \{x \in X : U \cap A \notin \mathcal{I},$ for every $U \in \tau^a(x)\}$ and $\Re_a(A) = X \setminus (X \setminus A)^{a^*} = \{x \in X :$ there exists $U_x \in \tau^a(x)$ such that $U_x \setminus A \in \mathcal{I}\}$. This $()^{a^*}$ operator gives a topology [2, 1] and it is denoted as τ^{a^*}, where $\beta(\mathcal{I},\tau) = \{V \setminus J : V \in \tau^a,\ J \in \mathcal{I}\}$ is a basis of it. We will denote 'Int^{a^*}' and 'Cl^{a^*}' as 'interior' operator and 'closure' operator of (X,τ^{a^*}) respectively.

THEOREM 2.1. [2] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space, then the following properties are equivalent:*

1. $\tau^a \cap \mathcal{I} = \{\emptyset\}$;

2. *If* $I \in \mathcal{I}$, *then* $aInt(I) = \emptyset$;
3. *For every* $G \in \tau^a, G \subseteq G^{a^*}$;
4. $X = X^{a^*}$.

Now we prove a theorem on ideal topological space which is meaningful for this paper.

THEOREM 2.2. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* $O \in \tau^a$ *then* $\tau^a \cap \mathcal{I} = \{\emptyset\}$ *if and only if* $O^{a^*} = aCl(O)$.

Proof. Suppose $\tau^a \cap \mathcal{I} = \{\emptyset\}$. It is obvious that $O^{a^*} \subseteq aCl(O)$. For reverse inclusion, let $x \in aCl(O)$. Then for every $U_x \in \tau^a(x), U_x \cap O \neq \emptyset$. Implies that $U_x \cap O \notin \mathcal{I}$ (as $\mathcal{I} \cap \tau^a = \{\emptyset\}$). Thus $x \in O^{a^*}$.

Converse is obvious from Theorem 2.1. □

THEOREM 2.3. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space, then the following properties are equivalent:*

1. $\tau^a \cap \mathcal{I} = \{\emptyset\}$;
2. *If* $I \in \mathcal{I}$, *then* $Int(I) = \emptyset$;
3. *For every* $G \in \tau^a, G \subseteq G^{a^*}$;
4. $X = X^{a^*}$;
5. *If* $O \in \tau^a$, *then* $O^{a^*} = aCl(O)$.

Proof. Obvious from Theorem 2.1 and Theorem 2.2. □

DEFINITION 2.4. [1] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. An operator $\Re_a : \wp(X) \to \tau^a$ is defined as follows for every $A \in \wp(X)$, $\Re_a(A) = X \setminus (X \setminus A)^{a^*}$.

Equivalently, $\Re_a(A) = \{x \in X :$ there exists a $U \in \tau^a(x)$ such that $U \setminus A \in \mathcal{I}\}$.

For this section, following theorem is an important part.

THEOREM 2.5. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space.* $\tau^a \cap \mathcal{I} = \{\emptyset\}$ *if and only if* $(\Re_a(A))^{a^*} = aCl(\Re_a(A))$ *for every* $A \subseteq X$.

Proof. Let $\tau^a \cap \mathcal{I} = \{\emptyset\}$. It is obvious that $(\Re_a(A))^{a^*} \subseteq aCl(\Re_a(A))$. For reverse inclusion, let $x \in aCl(\Re_a(A))$. Then for every a-open set V_x containing x, $V_x \cap \Re_a(A) \neq \emptyset$, implies that $V_x \cap \Re_a(A) \notin \mathcal{I}$, since $\tau^a \cap \mathcal{I} = \{\emptyset\}$. Therefore $x \in (\Re_a(A))^{a^*}$. Hence $(\Re_a(A))^{a^*} = aCl(\Re_a(A))$. Conversely suppose that $(\Re_a(A))^{a^*} = aCl(\Re_a(A))$ for every $A \subseteq X$. Then putting $A = X$, the condition $(\Re_a(A))^{a^*} = aCl(\Re_a(A))$ gives $(X \setminus (X \setminus X)^{a^*})^{a^*} = aCl(X \setminus (X \setminus X)^{a^*})$ i.e., $X^{a^*} = aCl(X)$. So $X^{a^*} = X$, and hence $\tau^a \cap \mathcal{I} = \{\emptyset\}$ (using Theorem 2.3). □

Using this Theorem we get following:

THEOREM 2.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the following properties are equivalent:*

1. $\tau^a \cap \mathcal{I} = \{\emptyset\}$;
2. $\Re_a(\emptyset) = \emptyset$;
3. *If $A \subseteq X$ is closed, then* $\Re_a(A)^{a^*} \setminus A = \emptyset$;
4. *If $A \subseteq X$, then* $aInt(aCl(A)) = \Re_a(aInt(aCl(A)))$;
5. *A is regular open in (X, τ^a),* $A = \Re_a(A)$;
6. *If $U \in \tau^a$, then* $\Re_a(U) \subseteq aInt(aCl(U)) \subseteq U^{a^*}$;
7. *If $I \in \mathcal{I}$, then* $\Re_a(I) = \emptyset$;
8. $(\Re_a(A))^{a^*} = aCl(\Re_a(A))$, *for every $A \subseteq X$.*

Proof. Obvious from Theorem 2.1, Theorem 2.2, Theorem 2.3, and Theorem 2.5. □

3. ${\Re_a}^*$-sets

For representation of α-open sets and semi-open sets the following is an important set:

DEFINITION 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a ${\Re_a}^*$-set if $A \subseteq (\Re_a(A))^{a^*}$.

The collection of all ${\Re_a}^*$-sets in $(X, \tau, \mathcal{I})$ is denoted by ${\Re_a}^*(X, \tau^a)$.
Following example is the existence of ${\Re_a}^*$-set:

EXAMPLE 3.2. *Let $X = \{e, b, c, d\}$, $\tau = \{\emptyset, X, \{e\}, \{c\}, \{e, b\}, \{e, c\}, \{e, b, c\}, \{e, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$. Regular open sets are $\emptyset$, X, $\{c\}$, $\{e, b\}$. Then $\tau^a = \{\emptyset, X, \{c\}, \{e, b\}, \{e, b, c\}\}$. Now $\Re_a(\{e, c, d\}) = X \setminus (\{b\})^{a^*} = X$. Thus $\{e, c, d\} \subseteq (\Re_a(\{e, c, d\}))^{a^*}$.*

It is obvious that ${\Re_a}^*(X, \tau^a) \subseteq \Re_a(X, \tau^a)$, where $\Re_a(X, \tau^a) = \{A \subseteq X : A \subseteq aCl(\Re_a(A))\}$ [11].
The reverse inclusion need not hold:

EXAMPLE 3.3. *Let $X = \{e, b, c, d\}$, $\tau = \{\emptyset, X, \{e\}, \{c\}, \{e, b\}, \{e, c\}, \{e, b, c\}, \{e, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{c\}\}$. Regular open sets are $\emptyset$, X, $\{c\}$, $\{e, b\}$. Then $\tau^a = \{\emptyset, X, \{c\}, \{e, b\}, \{e, b, c\}\}$. Then $\Re_a(\{c, d\}) = X \setminus (\{e, b\})^{a^*} = X \setminus \{e, b, d\} = \{c\}$ and $(\Re_a(\{c, d\}))^{a^*} = \emptyset$. Thus $\{c, d\} \notin {\Re_a}^*(X, \tau^a)$. Again $aCl(\{c\}) = \{c, d\}$. Hence $\{c, d\} \in \Re_a(X, \tau^a)$.*

We shall discuss the properties of ${\Re_a}^*$-sets:

THEOREM 3.4. *Let $\{A_i : i \in \triangle\}$ be a collection of nonempty ${\Re_a}^*$-sets in an ideal topological space $(X, \tau, \mathcal{I})$, then $\bigcup_{i\in\triangle} A_i \in {\Re_a}^*(X, \tau^a)$.*

Proof. For each $i \in \triangle$, $A_i \subseteq (\Re_a(A_i))^{a^*} \subseteq (\Re_a(\bigcup_{i\in\triangle} A_i))^{a^*}$. This implies $\bigcup_{i\in\triangle} A_i \subseteq (\Re_a(\bigcup_{i\in\triangle} A_i))^{a^*}$. Thus $\bigcup_{i\in\triangle} A_i \in {\Re_a}^*(X, \tau)$. □

Following example shows that the intersection of two ${\Re_a}^*$ sets in $(X, \tau, \mathcal{I})$ may not be a ${\Re_a}^*$-set.

EXAMPLE 3.5. *Let $X = \{e, b, c, d\}$, $\tau = \{\emptyset, X, \{e\}, \{c\}, \{e, b\}, \{e, c\}, \{e, b, c\}, \{e, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$. Regular open sets are $\emptyset$, X, $\{c\}$, $\{e, b\}$. Then $\tau^a = \{\emptyset, X, \{c\}, \{e, b\}, \{e, b, c\}\}$. Therefore $(\Re_a(\{c, d\}))^{a^*} = (X \setminus (\{e, b\})^{a^*})^{a^*} = (X\setminus\{e, b, d\})^{a^*} = (\{c\})^{a^*} = \{c, d\}$ and $(\Re_a(\{e, b, d\}))^{a^*} = (X\setminus(\{c\})^{a^*})^{a^*} = (X\setminus\{c, d\})^{a^*} = (\{e, b\})^{a^*} = \{e, b, d\}$. Now $(\Re_a(\{d\}))^{a^*} = (X \setminus (\{e, b, c\})^{a^*})^{a^*} = (X \setminus \{e, b, c, d\})^{a^*} = \emptyset$. Hence we have $\{c, d\}$ and $\{e, b, d\}$ are ${\Re_a}^*$ sets but their intersection $\{d\}$ is not a ${\Re_a}^*$-set.*

THEOREM 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $A \in \tau^{a^*}$ then $A \in {\Re_a}^*(X, \tau^a)$, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$.*

Proof. Given that $A \in \tau^{a^*}$. Then $Int^{a^*}(A) = A \cap \Re_a(A)$ [1]. Again $\Re_a(A) \in \tau^a$ and $\tau^a \cap \mathcal{I} = \{\emptyset\}$ implies that $(\Re_a(A))^{a^*} = aCl(\Re_a(A))$. Thus $A \subseteq aCl(\Re_a(A))$. Thus $A \in {\Re_a}^*(X, \tau^a)$. □

COROLLARY 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $A \in \tau^a$ then $A \in {\Re_a}^*(X, \tau^a)$, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$.*

Proof. Proof is obvious from the fact that $\tau^a \subseteq \tau^{a^*}$. □

From Example 3.5, we have intersection of two ${\Re_a}^*$-sets need not be a ${\Re_a}^*$-set in general. But following hold:

THEOREM 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \in {\Re_a}^*(X, \tau^a)$. If $U \in \tau^{a^\alpha}$, then $U \cap A \in {\Re_a}^*(X, \tau^a)$, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$.*

Proof. $U \cap A \subseteq aInt(aCl(aInt(U))) \cap (\Re_a(A))^{a^*} \subseteq (aInt(aCl(\Re_a(U)))) \cap (\Re_a(A))^{a^*} \subseteq (aInt(aCl(\Re_a(U)))) \cap (aCl(\Re_a(A)))$. Since $(aInt(aCl(\Re_a(U))))$ is a-open, then $U \cap A \subseteq aCl[aInt((aCl(\Re_a(U))) \cap (\Re_a(A)))] \subseteq aCl[aInt[aCl(\Re_a(U)) \cap \Re_a(A)]]$ as $\Re_a(A)$ is a-open set. This implies $U \cap A \subseteq aCl[aInt[aCl[\Re_a(U) \cap \Re_a(A)]]] = aCl[aInt[aCl(\Re_a(U \cap A))]] \subseteq aCl(\Re_a(U \cap A)) = (\Re_a(U \cap A))^{a^*}$. Thus $U \cap A \in {\Re_a}^*(X, \tau^a)$. □

4. $\dot{\Re}^*_a$-sets

Through this section we give the representation of α-sets and semi-open sets:

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a $\dot{\Re}^*_a$-set if $A \subseteq aInt(\Re_a(A))^*$.

The collection of all $\dot{\Re}^*_a$-set in $(X, \tau, \mathcal{I})$ is denoted by $\dot{\Re}^*_a(X, \tau^a)$.

The collection $\Re_a{}^*(X, \tau^a)$ does not form a topology, but the collection $\dot{\Re}^*_a(X, \tau^a)$ forms a topology.

At first we give an Example, which is the existence of $\dot{\Re}^*_a$-set.

Example 4.2. *Let $X = \{e, b, c, d\}$, $\tau = \{\emptyset, X, \{e\}, \{c\}, \{e, b\}, \{e, c\}, \{e, b, c\}, \{e, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{c\}\}$. Regular open sets are: $\emptyset$, X, $\{c\}$, $\{e, b\}$. Then $\tau^a = \{\emptyset, X, \{c\}, \{e, b\}, \{e, b, c\}\}$. Now $\Re_a(\{e, b\}) = X \setminus (\{c, d\})^{a^*} = X \setminus \{d\} = \{e, b, c\}$. Therefore $aInt(\Re_a(\{e, b\}))^{a^*} = aInt(\{e, b, d\}) = \{e, b\}$. Hence $\{e, b\} \in \dot{\Re}^*_a(X, \tau^a)$.*

It is obvious that $\dot{\Re}^*_a(X, \tau^a) \subseteq \Re_a{}^*(X, \tau^a)$ and $\dot{\Re}^*_a(X, \tau^a) \subseteq \tau^{a^{\Re_a}}$ hold, where $\tau^{a^{\Re_a}} = \{A \subseteq X : A \subseteq aInt(aCl(\Re_a(A)))\}$ [11]. For reverse direction we discuss following:

Example 4.3. *Let $X = \{e, b, c, d\}$, $\tau = \{\emptyset, X, \{e\}, \{c\}, \{e, b\}, \{e, c\}, \{e, b, c\}, \{e, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$. Regular open sets are: $\emptyset$, X, $\{c\}$, $\{e, b\}$. Then $\tau^a = \{\emptyset, X, \{c\}, \{e, b\}, \{e, b, c\}\}$. Then $(\Re_a(\{e, b, d\}))^{a^*} = (X \setminus (\{c\})^{a^*})^{a^*} = (X \setminus \{c, d\})^{a^*} = (\{e, b\})^{a^*} = \{e, b, d\}$ and $aInt(\Re_a(\{e, b, d\}))^{a^*} = \{e, b\}$. Thus $\{e, b, d\} \in \Re_a{}^*(X, \tau^a)$ but $\{e, b, d\} \notin \dot{\Re}^*_a(X, \tau^a)$.*

Example 4.4. *Let $X = \{e, b, c, d\}$, $\tau = \{\emptyset, X, \{e\}, \{c\}, \{e, b\}, \{e, c\}, \{e, b, c\}, \{e, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{c\}\}$. Regular open sets are: $\emptyset$, X, $\{c\}$, $\{e, b\}$. Then $\tau^a = \{\emptyset, X, \{c\}, \{e, b\}, \{e, b, c\}\}$. Then $aInt(\Re_a(\{c\}))^{a^*} = (X \setminus (\{e, b, d\})^{a^*})^{a^*} = (X \setminus \{e, b, d\})^{a^*} = (\{c\})^{a^*} = \emptyset$. Again $aInt(aCl(\{c\})) = \{c\}$. Thus $\{c\} \in \tau^{a^{\Re_a}}$ but $\{c\} \notin \dot{\Re}^*_a(X, \tau^a)$.*

Theorem 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then the collection $\dot{\Re}^*_a(X, \tau^a) = \{A \subseteq X : A \subseteq aInt(\Re_a(A))^{a^*}\}$ forms a topology on X, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$.*

Proof. (i) From Theorem 2.6, it is obvious that $\emptyset$, $X \in \dot{\Re}^*_a(X, \tau^a)$.

(ii) Let $A_i \in \dot{\Re}^*_a(X, \tau^a)$ for all i. Now we are to show that $\bigcup_i A_i \in \dot{\Re}^*_a(X, \tau^a)$. Since $A_i \subseteq \bigcup_i A_i$, $\Re_a(A_i) \subseteq \Re_a(\bigcup_i A_i)$ [1]. Thus $(\Re_a(A_i))^{a^*} \subseteq$

$(\Re_a(\bigcup_i A_i))^{a^*}$. So $A_i \subseteq aInt(\Re_a(A_i))^{a^*} \subseteq aInt(\Re_a(\bigcup_i A_i))^{a^*}$. Therefore $\bigcup_i A_i \in \dot{\Re}_a^*(X, \tau^a)$.

(iii) Let $A_1,\ A_2 \in \dot{\Re}_a^*(X, \tau^a)$. We are to show that $A_1 \cap A_2 \in \dot{\Re}_a^*(X, \tau^a)$. If $A_1 \cap A_2 = \emptyset$, we are done. Let $A_1 \cap A_2 \neq \emptyset$. Let $x \in A_1 \cap A_2$. Now $A_1 \subseteq aInt(\Re_a(A_1)^{a^*})$ and $A_2 \subseteq aInt(\Re_a(A_2)^{a^*})$, implies that $x \in aInt(\Re_a(A_1))^{a^*} \cap aInt(\Re_a(A_2))^{a^*}$. So $x \in aInt[(\Re_a(A_1))^{a^*} \cap (\Re_a(A_2))^{a^*}]$. Therefore there exists an a-open set V_x containing x such that $V_x \subseteq [(\Re_a(A_1))^{a^*} \cap (\Re_a(A_2))^{a^*}]$. Let U_x be any a-open set containing x of (X, τ^a). Then $\emptyset \neq V_x \cap U_x \subseteq aCl(\Re_a(A_1))$ (by Theorem 2.5) and $V_x \cap U_x \subseteq aCl(\Re_a(A_2))$ (by Theorem 2.5). Let $y \in V_x \cap U_x$. Consider any a-open set G_y containing y. Without loss of generality we may suppose that $G_y \subseteq V_x \cap U_x$. So $G_y \cap (\Re_a(A_1)) \neq \emptyset$. From definition of $\Re_a(A_1)$, there exists a $U \in \tau^a(x)$ such that $U \subseteq G_y$ and $U \setminus A_1 \in \mathcal{I}$. Again $U \subseteq aCl(\Re_a(A_2))$, so there exists a nonempty open set $U' \subseteq U$ such that $U' \setminus A_2 \in \mathcal{I}$. Now $U' \setminus (A_1 \cap A_2) = (U' \setminus A_1) \cup (U' \setminus A_2) \subseteq (U \setminus A_1) \cup (U' \setminus A_2) \in \mathcal{I}$. Hence from definition of $\Re_a$, $U' \subseteq \Re_a(A_1 \cap A_2)$. Since $U' \subseteq G_y$, $G_y \cap \Re_a(A_1 \cap A_2)$. Therefore $y \in aCl(\Re_a(A_1 \cap A_2))$. Since y was any point of $V_x \cap U_x$, it follows that $V_x \cap U_x \subseteq aCl(\Re_a(A_1 \cap A_2))$, implies that $x \in aInt(aCl(\Re_a(A_1 \cap A_2))) = aInt[(\Re_a(A_1 \cap A_2))^{a^*}]$. Thus $A_1 \cap A_2 \subseteq aInt[(\Re_a(A_1 \cap A_2))^{a^*}]$. Hence $A_1 \cap A_2 \in \dot{\Re}_a^*(X, \tau^a)$.

From (i), (ii) and (iii) $\dot{\Re}_a^*(X, \tau^a)$ forms a topology. □

For further discussion we mention following:

Proposition 4.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau^a \cap \mathcal{I} = \{\emptyset\}$. Then $\Re_a(A) \neq \emptyset$ if and only if A contains a nonempty τ^{a^*}-interior.*

Here we get two corollaries from above Proposition:

Corollary 4.7. *Let $x \in X$. Then $\{x\} \in {\Re_a}^*(X, \tau^a)$ if and only if $\{x\}$ is open in X with respect to the topology $\dot{\Re}_a^*(X, \tau^a)$.*

Proof. Let $\{x\} \in {\Re_a}^*(X, \tau^a)$ then $\Re_a(\{x\}) \neq \emptyset$. By Proposition 4.6, $\{x\}$ contains a nonempty τ^{a^*}-interior. Therefore $\{x\}$ is open in $\dot{\Re}_a^*(X, \tau^a)$. Conversely suppose that $\{x\}$ is open in $\dot{\Re}_a^*(X, \tau^a)$. Then $\{x\} \subseteq \Re_a(\{x\})$ [1]. Therefore $\{x\} \subseteq (\Re_a(\{x\})^{a^*}$, since $\tau^a \cap \mathcal{I} = \{\emptyset\}$ and $\Re_a(\{x\})$ is open in (X, τ^a), that is $\{x\} \in {\Re_a}^*(X, \tau^a)$. □

Corollary 4.8. *Let $x \in X$. Then $\{x\} \in {\Re_a}^*(X, \tau^a)$ if and only if $\{x\} \in \dot{\Re}_a^*(X, \tau^a)$.*

Proof. Let $\{x\} \in {\Re_a}^*(X, \tau^a)$, therefore $\{x\}$ is open in (X, τ^a) (by above corollary). Since $\{x\} \subseteq \Re_a(\{x\})$ [1], $\{x\} \subseteq aInt(aCl(\Re_a(\{x\})))$.

Thus $\{x\} \subseteq aInt(\Re_a(\{x\}))^{a^*}$ (from Theorem 2.5), and hence $\{x\} \in \dot{\Re}_a^*(X, \tau^a)$.

Conversely suppose that $\{x\} \in \dot{\Re}_a^*(X, \tau^a)$, then $\{x\} \subseteq aInt(\Re_a(\{x\}))^{a^*}$, implies that $\{x\} \subseteq (\Re_a(\{x\}))^{a^*}$, hence $\{x\} \in {\Re_a}^*(X, \tau^a)$. □

Following discussion are the representation for Njåstad's α-set of (X, τ^a):

THEOREM 4.9. *$\dot{\Re}_a^*(X, \tau^a)$ is the exactly the collection such that A belongs to $\dot{\Re}_a^*(X, \tau^a)$ and B belongs to ${\Re_a}^*(X, \tau^a)$ implies $A \cap B$ belongs to ${\Re_a}^*(X, \tau^a)$, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$.*

Proof. Let $A \in \dot{\Re}_a^*(X, \tau^a)$ and $B \in {\Re_a}^*(X, \tau^a)$. Now we are to show that $A \cap B \in {\Re_a}^*(X, \tau^a)$. If $A \cap B = \emptyset$, we are done. Let $A \cap B \neq \emptyset$. Let $x \in A \cap B$. This implies that $x \in aInt(\Re_a(A))^{a^*}$, therefore $x \in (\Re_a(A))^{a^*}$, and hence $x \in aCl(\Re_a(A))$. So for every a-open U_x containing x, $U_x \cap \Re_a(A) \neq \emptyset$. Again $x \in B \subseteq (\Re_a(B))^{a^*}$, then for every a-open set V_x containing x, $V_x \cap (\Re_a(B))^{a^*} \neq \emptyset$. Therefore for a-open set $W_x = U_x \cap V_x$ containing x, $W_x \cap \Re_a(A) \neq \emptyset$ and $W_x \cap \Re_a(B) \neq \emptyset$. Again $W_x \cap \Re_a(A) \subseteq W_x$ and $W_x \cap \Re_a(B) \subseteq W_x$. Therefore $W_x \cap [\Re_a(A) \cap \Re_a(B)] \neq \emptyset$. So $x \in aCl(\Re_a(A) \cap \Re_a(B))$, that is, $x \in aCl(\Re_a(A \cap B))$ [1]. Hence $A \cap B \subseteq aCl(\Re_a(A \cap B))$, therefore $A \cap B \in {\Re_a}^*(X, \tau^a)$. Next we consider a subset A of X such that $A \cap B \in {\Re_a}^*(X, \tau^a)$ for each $B \in {\Re_a}^*(X, \tau^a)$. We show that $A \in \dot{\Re}_a^*(X, \tau^a)$, that is, $A \subseteq aInt(\Re_a(A))^{a^*}$. If possible suppose that $x \in A$ but $x \notin aInt(\Re_a(A)^{a^*})$. Therefore $x \in A \cap (X \setminus aInt(\Re_a(A))^{a^*}) = A \cap aCl(X \setminus \Re_a(A)^{a^*}) = A \cap aCl(C)$ (say), where $C = X \setminus \Re_a(A)^{a^*}$. It is obvious that C is a nonempty a-open set, since $(\Re_a(A))^{a^*}$ is a nonempty a-closed set. Since $x \in aCl(C)$ then for all open set V_x containing x, $V_x \cap C \neq \emptyset$. Therefore $V_x \cap \Re_a(C) \neq \emptyset$, since $C \subseteq \Re_a(C)$. This implies that

$$x \in aCl(\Re_a(C)) \subseteq aCl(\Re_a(\{x\} \cup C)) \tag{4.1}$$

Again

$$C \subseteq aCl(\Re_a(C)) \subseteq aCl(\Re_a(\{x\} \cup C)) \tag{4.2}$$

From (4.1) and (4.2), $\{x\} \cup C \subseteq aCl(\Re_a(\{x\} \cup C))$. Therefore $\{x\} \cup C \in {\Re_a}^*(X, \tau^a)$. Now by hypothesis $A \cap (\{x\} \cup C)$ is a $\dot{\Re}_a^*$. If possible suppose that there exists $y \in X$ and $x \neq y$ such that $y \in A \cap (\{x\} \cup C)$. So $y \in A$ and $y \in C$. Now $A = A \cap X$ and $X \in {\Re_a}^*(X, \tau^a)$, again by hypothesis $A \in {\Re_a}^*(X, \tau^a)$. Since $y \in A$, $y \in (\Re_a(A))^{a^*}$, a contradiction to the fact that $y \in C = X \setminus \Re_a(A)^{a^*}$. Thus $A \cap (\{x\} \cup C) = \{x\}$. Since $\{x\} \in {\Re_a}^*(X, \tau^a)$, then $\{x\} \in \dot{\Re}_a^*(X, \tau^a)$. So $\{x\} \subseteq aInt(\Re_a(\{x\}))^{a^*} =$

$aInt[(\Re_a(A\cap(\{x\}\cup C)))^{a^*}] \subseteq aInt((\Re_a(A))^{a^*})$. So $x \in aInt((\Re_a(A))^{a^*})$, a contradiction to the fact that $x \notin aInt((\Re_a(A))^{a^*})$. Therefore we get $A \subseteq aInt((\Re_a(A))^{a^*})$, that is, $A \in \dot{\Re}_a^*(X, \tau^a)$. □

THEOREM 4.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$. Then $SO(X, \tau^{a^*}) = \{A \subseteq X : A \subseteq (\Re_a(A))^{a^*}\} = {\Re_a}^*(X, \tau^a)$.*

Proof. Let $A \in SO(X, \tau^{a^*})$. Then $A \subseteq Cl^{a^*}(Int^{a^*}(A)) = Cl^{a^*}(A \cap \Re_a(A))$[1]$\subseteq Cl^{a^*}(\Re_a(A)) = \Re_a(A) \cup (\Re_a(A))^{a^*} = \Re_a(A))^{a^*}$ (by Theorem 2.1). Hence $A \in {\Re_a}^*(X, \tau^a)$. So $SO(X, \tau^{a^*}) \subseteq {\Re_a}^*(X, \tau^a)$.

For reverse inclusion, let $A \in {\Re_a}^*(X, \tau^a)$. We show that $A \in SO(X, \tau^{a^*})$. Given that $A \subseteq (\Re_a(A))^{a^*}$, and $A \subseteq A$, then $A \subseteq A\cap(\Re_a(A))^{a^*}$. So $A \subseteq A\cap aCl(\Re_a(A)) \subseteq aCl[A\cap(\Re_a(A))] = aCl(Int^{a^*}(A)) \subseteq aCl[(Int^{a^*}(A))\cup (Int^{a^*}(A))^{a^*}] = aCl(Cl^{a^*}(Int^{a^*}(A))) = Cl^{a^*}(Int^{a*}(A))$, since $Cl^{a^*}(Int^{a^*}(A))$ is a-closed in (X, τ^a). Therefore $A \in SO(X, \tau^{a^*})$. Thus ${\Re_a}^*(X, \tau^a) \subseteq SO(X, \tau^{a^*})$. Hence we get the result. □

This Theorem is the representation theorem of Levine's semi-open sets.

REMARK 4.11. Let $x \in X$, then $\{x\} \in SO(X, \tau^{a^*})$ if and only if $\{x\} \in \dot{\Re}_a^*(X, \tau^a)$.

Proof. Proof is obvious from Corollary 4.7. □

THEOREM 4.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. $\dot{\Re}_a^*(X, \tau^a)$ is exactly the collection such that A belongs to $\dot{\Re}_a^*(X, \tau^a)$ and B belongs to $SO(X, \tau^{a^*})$ implies $A \cap B$ belongs to $SO(X, \tau^{a^*})$, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$*

Proof. Proof is obvious from Theorem 4.9 and 4.10. □

Now we shall discuss the relation between τ^{a^α} with $\dot{\Re}_a^*(X, \tau^a)$. For this we mention a Theorem owning to O. Njåstad:

THEOREM 4.13. [15] *Let (X, τ) be a topological space. τ^α consists of exactly those sets for which $A \cap B \in SO(X, \tau)$ for all $B \in SO(X, \tau)$.*

From above two Theorems we have:

THEOREM 4.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$. Then $\dot{\Re}_a^*(X, \tau^a) = \tau^{{a^*}^\alpha}$.*

This Theorem is the representation theorem of Njåstad's α-sets.

Further we get following remark:

REMARK 4.15. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\tau^a \cap \mathcal{I} = \{\emptyset\}$. Then $\dot{\Re}_a^*(X, \tau^a) = (\tau^{a^{*\alpha}}) = \tau^{a^{\Re_a}}$.

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